KEELE UNIVERSITY

SCHOOL OF PHYSICAL AND GEOGRAPHICAL SCIENCES

Year 1 ASTROPHYSICS LAB

SOLUTIONS to Week 1 Exercises

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1. Parallax and Angular Distances

QUESTION 1

Figure 1 in the lab script shows that, **if the parallax p is exactly one arcsecond** ($p = 1''$), then

$$\tan p \;=\; \tan(1'') \;=\; 1 \text{ AU}/1 \text{ pc}$$

In the **small-angle approximation** (which will always be valid for anything we do in this Lab), $\tan p \simeq p$ if p is in **radians**. Thus, we need to convert 1 arcsec to the equivalent number of radians, and then set this equal to 1 AU $\div$ 1 pc. That is,

$$\begin{aligned} 1 \text{ arcsec} &= 1 \text{ arcsec} \times \frac{1 \text{ degree}}{3600 \text{ arcsec}} \times \frac{2\pi \text{ radians}}{360 \text{ degrees}} \\ &= 4.84814 \times 10^{-6} \text{ rad} \\ &= \frac{1}{206,265} \text{ rad} \end{aligned}$$

and therefore

$$\tan(1'') \;=\; \tan\left(\frac{1}{206,265}\text{ rad}\right) \;\simeq\; \frac{1}{206,265} \;=\; \frac{1 \text{ AU}}{1 \text{ pc}}$$

Solving for the value of 1 pc now gives

$$1 \text{ pc} \;=\; 206,265 \text{ AU} \;=\; 3.094 \times 10^{16} \text{ m}$$

where the second equality follows from the fact that 1 AU $= 1.5 \times 10^{11}$ m. Finally, one light year is equal to

$$\begin{aligned} 1 \text{ l.y.} &= 3 \times 10^{8} \text{ m s}^{-1} \times (3600 \text{ s/hour} \times 24 \text{ hour/day} \times 365.25 \text{ day/year}) \\ &= 9.467 \times 10^{15} \text{ m} \end{aligned}$$

so we also have

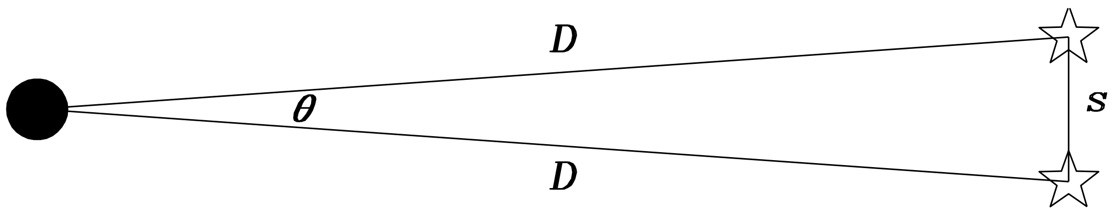
$$1 \text{ pc} \;=\; 3.26 \text{ light year}$$

Question 2

By the definition of pc, if the trigonometric parallax p is in arcsec and the distance D is in parsec, then $p = 1/D$. So, for $D = 6$ pc,

$$p = \frac{1}{6} \text{ arcsec}$$

Question 3



The Sun is out of the picture now. The idea is not to observe one distant object at two times, when the Earth is on opposite sides of the Sun, but to observe two equidistant objects simultaneously, from a single point on the Earth's orbit around the Sun. As such, the angular separation of the two distant objects (labelled θ in the diagram) is not called a parallax, and so to avoid confusion it is not referred to by the symbol p. Nevertheless, the formula linking θ, s, and D is in essence the same as the formula for trigonometric parallax.

From the diagram, it follows that $\sin(\theta/2) = (s/2)/D$. Assuming that $s \ll D$, as the diagram suggests, it must be that θ is a very small angle, so the approximation $\sin(\theta/2) \simeq \theta/2$ is a good one (if θ is in **radians**). Thus,

$$\theta = \frac{s}{D} \qquad [\theta \text{ in radians}] \tag{1}$$

If we want to work with θ in arcseconds instead, then we make use of the fact that there are 206,265 arcsec in one radian (see Question 1), so

$$\theta = 206,265 \, \frac{s}{D} \qquad [\theta \text{ in arcsec}] \tag{2}$$

QUESTION 4

For an angular separation $\theta = 206''$ and a distance $D = 2$ AU to both objects, equation (2) in Question 3 gives their linear separation s as

$$\begin{aligned} s &= \frac{\theta\, D}{206,265} = \frac{206 \times 2 \text{ AU}}{206,265} \\ &= 0.00197 \text{ AU} \;=\; 3.00 \times 10^5 \text{ km} \end{aligned}$$

If the objects are 2 AU from the Earth in the direction away from the Sun, then they are at a distance of 3 AU from the Sun, which would place them in the asteroid belt between Mars and Jupiter.

QUESTION 5

To obtain the angular distance between the galaxies in arcsec, use equation (2) from Question 3 again:

$$\theta \;=\; 206,265\,\frac{s}{D} \;=\; 206,265 \;\times\; \frac{8 \times 10^3 \text{ pc}}{20 \times 10^6 \text{ pc}} \;=\; 82.5 \text{ arcsec}$$

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2. Celestial Coordinates

QUESTION 6

Right ascension (RA, or α) and declination (Dec, or δ) are longitude and latitude in the system of celestial coordinates defined by “projecting” the Earth’s equator onto the sky.

RA is the longitude coordinate. It is measured in hours (h), minutes (m), and seconds (s), where 24 hours is a complete circle (360°), so $1^{\text{h}} = 15°$ (like time zones on Earth). There are 60 seconds in one minute, and 60 minutes in one hour of RA.

Dec is the latitude coordinate. It is measured in degrees (°), arcminutes ($'$), and arcseconds ($''$), where there are 60 arcsec in 1 arcmin and 60 arcmin in 1 degree. The celestial equator corresponds to $\delta = 0°$; the celestial north pole, to $\delta = +90°$; and the celestial south pole, to $\delta = -90°$.

QUESTION 7

Subtracting RA's is like subtracting time in hours, minutes, and seconds. Thus, the separation of the two points given is

$$\Delta\,\alpha \;=\; 11^{\mathrm{m}}56^{\mathrm{s}} \;=\; \frac{11}{60}\,\text{hour} + \frac{56}{3600}\,\text{hour} \;=\; 0.19888\ \text{hour}$$

Then, since one hour of RA is equal to 15°,

$$\Delta\,\alpha \;=\; 0.19888 \times 15 \;=\; 2.983\ \text{degrees}$$

QUESTION 8

The difference between the two Dec's given is

$$\Delta\,\delta \;=\; 68'45'' \;=\; (68 \times 60)\ \text{arcsec} + 45\ \text{arcsec} \;=\; 4125\ \text{arcsec}$$

Thus, from equation (2) in Question 3 above, the linear separation of the two galaxies on the plane of the sky is

$$\begin{aligned} s &= \frac{\theta\,D}{206,265} \qquad [\theta \text{ in arcsec}] \\ &= \frac{4125 \times 100\ \text{Mpc}}{206,265} \\ &= 2.00\ \text{Mpc} \end{aligned}$$

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3. Magnitude and Distance Modulus

QUESTION 9

Start by considering any two absolute magnitudes, M_1 and M_2, in the same filter (at the same wavelength) for any two stars. Since both magnitudes are absolute, it is as if both stars are at the same distance $D = 10$ pc. Therefore,

$$\begin{aligned} M_1 &= \mathit{const.} - 2.5\ \log\left(L_1/4\pi[10\ \text{pc}]^2\right) \\ M_2 &= \mathit{const.} - 2.5\ \log\left(L_2/4\pi[10\ \text{pc}]^2\right) \end{aligned}$$

The constant is the same in both equations if the magnitudes are in the same filter, so the constant cancels when the two magnitudes are subtracted:

$$M_1 - M_2 \;=\; -\,2.5 \times \left\{\log\left(L_1/4\pi[10\ \text{pc}]^2\right) - \log\left(L_2/4\pi[10\ \text{pc}]^2\right)\right\}$$

Now, subtracting the logarithm of two numbers is equivalent to taking the logarithm of the ratio of the numbers, so

$$M_1 - M_2 \;=\; -\,2.5 \times \log\left(\frac{L_1/4\pi[10\text{ pc}]^2}{L_2/4\pi[10\text{ pc}]^2}\right) \;=\; -\,2.5\,\log\left(\frac{L_1}{L_2}\right)$$

Notice how the distance of 10 pc has cancelled out of the magnitude difference, so that **the difference between the absolute magnitudes of two objects depends only on the ratio of the intrinsic luminosities**. Similarly, **the difference between the apparent magnitudes of two objects at any distance D (10 pc or othwerwise) does not depend on the value of D**, so long as it is the **same** for both objects. All of this holds in any particular filter/at any given wavelength.

The last equation above can now be used to solve for the ratio of the intrinsic luminosities of stars 1 and 2. In general,

$$M_1 - M_2 \;=\; -\,2.5\,\log\left(\frac{L_1}{L_2}\right) \qquad \Longleftrightarrow \qquad \frac{L_1}{L_2} \;=\; 10^{-(M_1-M_2)/2.5} \tag{3}$$

Therefore, in this specific example, take $M_{V,1} = +4.83$ and $M_{V,2} = +6.24$ to find

$$L_1/L_2 \;=\; 10^{-(M_{V,1}-M_{V,2})/2.5} \;=\; 10^{+1.41/2.5} \;=\; 3.664$$

Since $L_1/L_2 > 1$ here, the star with $M_{V,1} = +4.83$ is brighter than the star with $M_{V,2} = +6.24$. In general, **for objects at the same distance, lower magnitudes correspond to brighter intrinsic luminosities**. This is a result of the **minus sign** in the definition of magnitude.

Question 10

From equation (3) in Question (9), the absolute V magnitudes of any two stars with V-band luminosities L_1 and L_2 are related by

$$M_{V,1} - M_{V,2} \;=\; -\,2.5\,\log(L_1/L_2)$$

or, equivalently,

$$M_{V,2} \;=\; M_{V,1} + 2.5\,\log(L_1/L_2)$$

Let star 1 be the Sun, so that $M_{V,1} = +4.83$. Then for the star that is 100 times **brighter** than the Sun, we have $L_2 = 100\,L_1$, and

$$M_{V,2} \;=\; 4.83 + 2.5\,\log(1/100) \;=\; -\,0.17$$

(which shows again that, at a fixed distance, brighter objects have lower magnitudes).

For the star that is 100 times **fainter** than the Sun, we have $L_2 = 0.01\,L_1$, so

$$M_{V,2} \;=\; 4.83 + 2.5\,\log(1/0.01) \;=\; +9.83$$

Question 11

Returning to the definition of magnitude, for distance D measured in pc the apparent and absolute magnitudes of an object with luminosity L are

$$\begin{aligned} m &= const. - 2.5\,\log\left(L/4\pi D^2\right) \\ M &= const. - 2.5\,\log\left(L/4\pi[10]^2\right) \end{aligned}$$

Subtracting these equations to cancel the wavelength/filter-dependent constant gives the **distance modulus**:

$$\begin{aligned} m - M &= -2.5\,\log\left(L/4\pi D^2\right) + 2.5\,\log\left(L/4\pi[10]^2\right) \\ &= 2.5 \times \left\{\log\left(L/4\pi[10]^2\right) - \log\left(L/4\pi D^2\right)\right\} \\ &= 2.5\,\log\left(\frac{L/4\pi[10^2]}{L/4\pi D^2}\right) \\ &= 2.5\,\log\left(\frac{D^2}{10^2}\right) \\ &= 2.5\,\log(D^2) - 2.5\,\log(10^2) \\ &= 5\,\log D - 5 \end{aligned}$$

where the last line uses the rule, $\log(x^2) = 2\,\log(x)$ (and the fact that $\log 10 = 1$).

Notice how the luminosity L and the constant 4π both cancelled out of the calculation.

Question 12

The equation from Question (11) for distance modulus can be solved for the apparent magnitude of an object at **distance D in parsecs**:

$$m \;=\; M + 5\,\log D - 5$$

in general.

A star with the luminosity of the Sun has an absolute magnitude in the V band of $M_V = +4.83$ (Question 10). Thus, its apparent V magnitude at a distance of 5 pc is

$$D = 5 \text{ pc} \quad \Longrightarrow \quad V \;=\; 4.83 + 5\,\log(5) - 5 \;=\; +\,3.32$$

and its apparent magnitude at a distance of 50 pc is

$$D = 50 \text{ pc} \quad \Longrightarrow \quad V \;=\; 4.83 + 5\,\log(50) - 5 \;=\; +\,8.32$$

Notice how the star has a lower apparent magnitude (is apparently brighter) at the closer distance.

Question 13

The equation from Question (11) for distance modulus can be rearranged to solve for D **in parsecs**:

$$m - M \;=\; 5\,\log D - 5 \qquad \Longrightarrow \qquad D \;=\; 10^{(m-M+5)/5}$$

Thus, an apparent magnitude $V = +11$ and an absolute magnitude $M_V = -20$ imply

$$D \;=\; 10^{(11+20+5)/5} \;=\; 10^{7.2} \;=\; 1.585 \times 10^7 \text{ pc} \;=\; 15.85 \text{ Mpc}$$

Question 14

The B-band apparent and absolute magnitudes of an object at distance D (in pc) are related by

$$B - M_B \;=\; 5\,\log D - 5$$

while the apparent and absolute V-band magnitudes of the same object are related by

$$V - M_V \;=\; 5\,\log D - 5$$

Subtracting these two equations **cancels the distance term**, leaving just

$$(B - M_B) - (V - M_V) \;=\; 0$$

This rearranges to

$$(B - V) - (M_B - M_V) = 0 \qquad \Longrightarrow \qquad B - V \;=\; M_B - M_V$$

That is, the apparent colour of an object is exactly equal to its absolute colour, no matter what the value of D. Put another way, **colour is independent of distance**. This is a general result, true of the colour index through any two filters.

For the star in question, then, $\boldsymbol{B - V = 1.0}$ **at all of the distances asked—and at any other**. Since the B-band magnitude is larger than the V-band magnitude at any distance, the star is intrinsically fainter in the B band.

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4. Measurement Uncertainties

QUESTION 15

If the trigonometric parallax p is given in units of arcsec, then the distance $D = 1/p$ in units of parsec, in which case the basic rule for error propagation through a function implies

$$D = 1/p \qquad \Longrightarrow \qquad \Delta D \;=\; \Delta p \times \left| \frac{d}{dp}\,(1/p) \right| \;=\; \frac{\Delta p}{p^2}$$

Therefore,

$$p \;=\; 0\overset{''}{.}100 \pm 0\overset{''}{.}003 \qquad \Longrightarrow \qquad D \;=\; 10.0 \pm 0.3 \;\; \text{pc}$$

QUESTION 16

Since magnitude is $m = const. - 2.5\,\log(L/4\pi D^2)$, in the case that there is an uncertainty in L we have

$$\begin{aligned}
\Delta m \;&=\; \Delta L \times \left| \frac{d}{dL}\left[const. - 2.5\,\log\,L + 2.5\,\log(4\pi D^2)\right] \right| \\
&=\; \Delta L \times \left| \frac{d}{dL}\,(-2.5\,\log\,L) \right| \;=\; \Delta L \times \frac{2.5}{(\ln 10) \times L} \\
&=\; \frac{2.5}{\ln 10}\,\frac{\Delta L}{L}
\end{aligned}$$

(since neither the constant nor D is a function of L, those two derivatives on the first line are zero).

Thus,

$$L \;=\; 10 \pm 1\,L_\odot \qquad \Longrightarrow \qquad \Delta m \;=\; \frac{2.5}{\ln 10}\,\frac{1\;L_\odot}{10\;L_\odot} \;\simeq\; 0.11 \;\; \text{mag}$$

Question 17

Rearranging the Doppler-shift equation to solve for V,

$$\frac{V}{c} = \frac{\lambda - \lambda_0}{\lambda_0} \qquad \Longrightarrow \qquad V = \frac{c(\lambda - \lambda_0)}{\lambda_0} = \frac{c}{\lambda_0}\,\lambda - c$$

so applying the rule of error propagation through a function gives

$$\Delta V = \Delta\lambda \times \left|\frac{dV}{d\lambda}\right| = \frac{c}{\lambda_0}\,\Delta\lambda$$

Thus, for $\lambda = 495$ nm, $\Delta\lambda = 15$ nm, $\lambda_0 = 450$ nm, and $c = 3 \times 10^8$ m s^{-1},

$$V = (3 \times 10^7) \pm (1 \times 10^7) \;\text{ m s}^{-1} = (3 \pm 1) \times 10^7 \;\text{ km s}^{-1}$$

Question 18

Given $a = 3.0 \pm 0.2$ m, the uncertainty in a^2 follows as

$$\begin{aligned}\Delta(a^2) &= \Delta a \times \left|\frac{d}{da}\,(a^2)\right| \\ &= \Delta a \times 2a \\ &= 0.2 \times (2 \times 3) = 1.2 \text{ m}^2\end{aligned}$$

Similarly, given $b = 4.0 \pm 0.3$ m, the uncertainty in b^2 is

$$\Delta(b^2) = 2b\,\Delta b = 2 \times 4 \times 0.3 = 2.4 \text{ m}^2$$

The uncertainty in the *square* of the hypotenuse then follows from the standard rule for combining the uncertainties of a sum:

$$\begin{aligned}\Delta(c^2) &= \sqrt{[\Delta(a^2)]^2 + [\Delta(b^2)]^2} \\ &= \sqrt{(1.2)^2 + (2.4)^2} = 2.6833 \text{ m}^2\end{aligned}$$

To this point, then, we have

$$c^2 = a^2 + b^2 = 25.0 \pm 2.6833 \text{ m}^2$$

The hypotenuse itself is, of course, the square root of c^2. In general, the uncertainty in the square root of any number X is (going back to the rule for error propagation through a function)

$$\Delta(X^{1/2}) = \Delta X \times \left|\frac{d}{dX}\,X^{1/2}\right| = \Delta X \times \frac{1}{2}\,X^{-1/2} = \frac{\Delta X}{2\,X^{1/2}}$$

Thus, making the association $X = c^2 = 25.0 \pm 2.6833$ m^2,

$$\Delta\, c \;=\; \frac{2.6833}{2 \times \sqrt{25.0}} \;=\; 0.26833 \text{ m}$$

so, paying attention to significant digits for the final answer,

$$c \;=\; 5.0 \pm 0.3 \text{ m}$$

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5. Excel Plotting and Fitting Straight Lines

Fitting a straight line of the basic form

$$V = m \times D + c$$

to velocity (V) versus distance (D) from the data provided should give you a best-fit slope of (note the units)

$$m = 69.2 \pm 3.2 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

and a best-fit intercept of (note the units)

$$c = 2.8 \pm 117.6 \text{ km s}^{-1}$$

which, given the large uncertainty, is entirely consistent with $c = 0$. The Excel formula to obtain the uncertainty in the slope, Δm, in Cell B45 of the spreadsheet is `=SQRT(J39/B41)`. The formula for the uncertainty in the intercept, Δc, in Cell B46 is `=SQRT(H39/B41)`.

Your graph should look something like this: